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SOME COMMENTS ON THE ENDOCHRONIC THEORY OF PLASTICITY

R.S. RIVLIN
Center for the Application of Mathematics, Lehigh University, Bethlehem, PA 18015, U.S.A.

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Abstract—The endochronic theory of plasticity advanced by Valanis in 1971 and 1975[1,2] is discussed critically. It is shown that the theory leads to a number of predictions which are not in accord with the observed behavior of metals. A further theory[5], while not open to these objections, does share with the earlier theory the advantages over conventional plasticity theory which were claimed for it.

1. INTRODUCTION

In two papers[1, 2] published in 1971, Valanis claimed to derive from thermodynamic considerations three-dimensional constitutive equations on the basis of which the behavior of plastic, or visco-plastic materials, subjected to small or large deformations could be modeled. His theory was most fully developed in the case when the material is rate-independent and the deformations are sufficiently small so that the classical infinitesimal strain tensor can be used. Accordingly, in the present paper the theory will be discussed in this context.

In[1, 2], Valanis claimed for this theory the merit, over most other phenomenological theories which purport to describe the elastic–plastic behavior of metals, that it avoids the introduction of a yield surface. The stress is expressed as a functional, of specific form, of the history of the strain—a single such expression being valid both under conditions of loading and unloading. It also expresses the stress explicitly as a functional of the total strain and avoids the separation of strain into elastic and plastic parts.

Since the thermodynamic content of Valanis's argument is questionable and is, in any case, replete with *ad hoc* assumptions, discussion of it is relegated to the Appendix. There the argument is presented in a manner which retains its *essential* features, but is stripped of irrelevancies. However, even when presented in this manner, it is evident that certain assumptions are unacceptable and others are of a quite arbitrary character unsupported by physical justification. Accordingly, in the main body of the paper we discuss the constitutive equation without relation to this background, purely from the point of view of its suitability as a model for such materials as metals.

The essential feature of the constitutive equation of Valanis is that it assumes the stress following some deformation history to be a linear isotropic functional of the strain increment history, the kernel in this functional being a scalar function of an *intrinsic time* defined by an isotropic scalar functional of the strain increment history. Valanis uses the term *endochronic theory* to describe a theory of this type. Valanis assumes a quite explicit form for the functional defining the intrinsic time. He justifies it by an alleged reasonableness, which is subjective, and by alleged agreement between the predictions of the theory and experiment. It is seen in Sections 2 and 6 that even if we take this agreement at its face value, it could not establish the validity of his assumed expression for the intrinsic time, since a wide variety of other expressions would provide equally good agreement.

In this paper we discuss the theory in a slightly broader context, in which the particular form for the intrinsic time used by Valanis is replaced by a somewhat more general one (see eqn 5.4 below). It will be seen that most of the criticisms which can be levelled against the theory of Valanis also apply to theories based on this more general definition of intrinsic time.

For rate-independent materials, the strain-history can necessarily be parametrized in terms of the path-length covered by the strain history in 9-dimensional strain space. The intrinsic time, whether defined in the manner adopted by Valanis, or in the more general manner of the present paper, can then be parametrized in terms of this strain path length. As a matter of convenience we adopt this course in the present paper, although this is not essential to the arguments presented.

In the case of one-dimensional deformations, discussed in Sections 2–4, this strain path length is taken as the intrinsic time. For simple shearing deformations this involves no loss of generality since the intrinsic time is necessarily some constant multiple of the strain path length. This is also the case for simple extension, provided that we make the assumption, which is made by Valanis, that Poisson's ratio† (defined as the ratio decremental transverse strain/incremental longitudinal strain) is constant.

In Section 2 we establish certain relations which could, at any rate in principle, be used to test the applicability of the class of constitutive equations considered to a particular material for one-dimensional deformations.

In Section 3 we discuss the relation between the incremental moduli for loading and unloading from a state in the plastic regime. It has been pointed out that the constitutive equation adopted by Valanis yields, in disagreement with experimental results for metals, a value for the unloading modulus which is very different from the initial modulus for infinitesimal strains from the undeformed state. We find that this is not necessarily the case in the context of the more general theory discussed in the present paper. In Section 4, we discuss the dissipation in infinitesimal unloading-loading and loading-unloading strain cycles starting at an arbitrary strain which has been reached by a monotonic loading. We find, as did Sandler [4] in a more restricted context, that the dissipation must necessarily be negative in either one or the other of these cycles and accordingly, the material will be unstable.

The three-dimensional theory is introduced in Section 5. It is shown in Section 6 that for deformations consisting of successive simple extensions and simple shears, carried out in discrete time intervals, any definition of intrinsic time in the broader class envisaged in the present paper can lead to the same prediction for the stress as that used by Valanis, if appropriate values are given to the adjustable constants. Accordingly, agreement between theory and experiment for deformations of this type cannot be used to establish the predictive value of Valanis's constitutuve equation for more general deformations. Attention is drawn in Section 8 to a further general criticism of constitutive equations based on the endochronic concept. It is shown that if a strain is reached by two paths which differ only very slightly, the associated stresses may be very different, i.e. the relation between stress and strain history is not continuous in the sense of the supremum norm.

In [5] Valanis presented a revised theory in order to meet the criticisms which had been levelled at the earlier theory. This is discussed briefly in Section 9. He again motivates the theory by thermodynamic arguments which are analogous to, and open to the same criticisms as, those used to motivate the earlier theory. Accordingly, we do not discuss them explicitly in the present paper. While this second theory is, in fact, not open to most of the criticisms to which the earlier theory is subject, it does not possess the two major and somewhat revolutionary advantages which were claimed for the latter—that it describes the elastic–plastic behavior of metals without the introduction of a yield surface and without the separation of the strain into elastic and plastic parts. It differs from previous theories of plasticity only in the manner in which strain-hardening affects the yield surface and the relation between stress and incremental plastic strain. No evidence is adduced to demonstrate that it provides a more accurate description of the actual behavior of metals than do other constitutive equations that have been suggested.

The criticisms in this paper of the theory advanced in [1, 2] do not, of course, necessarily apply to all possible theories based on the endochronic concept, particularly if the endochronic time is defined in terms of the plastic, rather than the total, strain.

2. ONE-DIMENSIONAL THEORY

We consider small uniaxial deformations of a rate-independent material. Let ε denote the strain. Let l be the length of the strain-path, thus:

$$dl = |d\varepsilon|, \tag{2.1}$$

with $l = 0$ in the undeformed state. We now consider a deformation in which $\varepsilon = 0$ initially and

†This assumption, which was criticized by Lee [3] is discussed in Section 7.

increases monotonically to ε_1, then decreases monotonically to ε_2, increases monotonically to ε_3, and so on, the final reversal taking place at $\varepsilon = \varepsilon_\mu$. We denote the final value of ε at which the stress is measured by E. We note that if μ is odd (even) the last reversal will be followed by a decrease (increase) in ε.

Let l_α be the value of l corresponding to the αth reversal of the strain, and let L be the value of l corresponding to the final value E. It is easily seen from (2.1) that

$$l_\alpha = 2\sum_{\beta=1}^{\alpha-1}(-1)^{\beta-1}\varepsilon_\beta + (-1)^{\alpha-1}\varepsilon_\alpha, \tag{2.2}$$

and

$$L = 2\sum_{\beta=1}^{\mu}(-1)^{\beta-1}\varepsilon_\beta + (-1)^{\mu} E. \tag{2.3}$$

We now regard ε as a function of l and make the assumption that the stress σ is given by

$$\sigma = \int_0^L f(L, l)\mathrm{d}\varepsilon(l), \tag{2.4}$$

where f is a positive function of the indicated arguments. Noting that for $l_{\alpha-1} < l < l_\alpha$,

$$\mathrm{d}\varepsilon(l) = \mathrm{d}l(\alpha \text{ odd}) \quad \text{and} \quad \mathrm{d}\varepsilon(l) = -\mathrm{d}l\ (\alpha \text{ even}), \tag{2.5}$$

we obtain from (2.4)

$$\sigma = 2\sum_{\alpha=1}^{\mu}(-1)^{\alpha-1} g(L, l_\alpha) + (-1)^{\mu} g(L, L), \tag{2.6}$$

where

$$g(L, l) = \int_0^l f(L, l)\mathrm{d}l. \tag{2.7}$$

We can rewrite (2.6) as

$$\sigma = \begin{cases} \sum_{\alpha=1}^{\mu}(-1)^{\alpha-1}\sigma(L, l_\alpha) + \sigma(L, L)\ (\mu \text{ even}) \\ \sum_{\alpha=1}^{\mu}(-1)^{\alpha-1}\sigma(L, l_\alpha)\ \ (\mu \text{ odd}), \end{cases} \tag{2.8}$$

where

$$\sigma(L, l_\alpha) = 2g(L, l_\alpha) - g(L, L),$$
$$\sigma(L, L) = g(L, L). \tag{2.9}$$

We note that $\sigma(L, l_\alpha)$ is the stress, when the length of the strain path is L, in a deformation in which the strain increases monotonically from zero to l_α and then decreases monotonically until the total length of the strain path is L. Also, $\sigma(L, L)$ is the stress, at strain path length L, in a deformation in which the strain increases monotonically from 0 to L.

From (2.8) it is easily seen that

$$\sigma(L, l_1, \ldots, l_\mu) - \sigma(L, l_1, \ldots, l_{\mu-1}) = (-1)^{\mu}[\sigma(L, L) - \sigma(L, l_\mu)], \tag{2.10}$$

where $\sigma(L, l_1, \ldots, l_\mu)$ denotes the stress, at strain path length L, resulting from a deformation with μ reversals of strain at path lengths $l_1, l_2, \ldots, l_\mu$.

From (2.10) or (2.8), we easily obtain the relations

$$\sigma(L, l_1, \ldots, l_\alpha) - \sigma(L, l_1, \ldots, l_{\alpha-2}) = (-1)^\alpha[\sigma(L, l_{\alpha-1}) - \sigma(L, l_\alpha)]. \tag{2.11}$$

Conversely, by giving α the values $3, 5, \ldots, \mu$ (odd) in (2.11) and adding the resulting equations, we can recover $(2.8)_2$ and, by giving α the values $2, 4, \ldots, \mu$ (even) and adding the resulting equations, we can recover $(2.8)_1$. In deriving the second of these results, we note that when $\alpha = 0$, $\sigma(L, l_\alpha) = \sigma(L, L)$.

As an example of the implications of eqns (2.10) and (2.11), we consider a deformation in which the strain increases from zero to ε_1 and then undergoes n cycles of deformation between strains ε_1 and $\varepsilon_2(\varepsilon_2 < \varepsilon_1)$. In this case

$$l_\alpha = \varepsilon_1 + (\alpha - 1)(\varepsilon_1 - \varepsilon_2), \quad L = \varepsilon_1 + 2n(\varepsilon_1 - \varepsilon_2). \tag{2.12}$$

Taking $\alpha = 2n + 1$ in (2.11), we obtain

$$\sigma(L, l_1, \ldots, l_{2n+1}) - \sigma(L, l_1, \ldots, l_{2n-1}) = \sigma(L, l_{2n+1}) - \sigma(L, l_{2n}). \tag{2.13}$$

We note that the σ's in (2.13) have the following interpretations: $\sigma(L, l_1, \ldots, l_{2n+1})$ is the stress at strain ε_1 resulting from an initial deformation to strain ε_1 followed by n cycles between strains ε_1 and $\varepsilon_2(\varepsilon_2 < \varepsilon_1)$; $\sigma(L, l_1, \ldots, l_{2n-1})$ is the stress at strain $\varepsilon_1 + 2(\varepsilon_1 - \varepsilon_2)$ resulting from an initial deformation to strain ε_1 followed by $n - 1$ cycles between strains ε_1 and ε_2 and then a further increase in strain to $\varepsilon_1 + 2(\varepsilon_1 - \varepsilon_2)$; $\sigma(L, l_{2n+1}) = \sigma(L, L)$ is the stress at strain $\varepsilon_1 + 2n(\varepsilon_1 - \varepsilon_2)$ resulting from a monotonic deformation to this strain; $\sigma(L, l_{2n})$ is the stress at strain $\varepsilon_1 + 2(n-1)(\varepsilon_1 - \varepsilon_2)$ resulting from a monotonic deformation to strain $\varepsilon_1 + (2n-1)(\varepsilon_1 - \varepsilon_2)$ followed by a decrease in the strain to $\varepsilon_1 + 2(n-1)(\varepsilon_1 - \varepsilon_2)$.

Suppose now that we do not assume that the stress is given by an expression of the form (2.4). As before, we consider that initially the strain increases and this increase is followed by μ reversals of the strain at strains $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\mu$. Then, at $l = L$, the stress must be a function of L and of $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\mu$, and, hence from (2.5), of $L, l_1, l_2, \ldots, l_\mu$. We write

$$\sigma = \sigma(L, l_1, \ldots, l_\mu). \tag{2.14}$$

We note that if $\mu = 1$, the relation (2.10) is satisfied identically. (We bear in mind that if $\mu = 1$, $\sigma(L, l_1, \ldots, l_{\mu-1}) = \sigma(L, L)$.) It is easily seen that if the function $\sigma(L, l_1) = \bar{\sigma}(L, l_1)$, say, is specified, and we take

$$f(L, l) = \frac{1}{2}\frac{\partial}{\partial l}[\bar{\sigma}(L, l)] \tag{2.15}$$

in (2.4), we obtain

$$\sigma(L, l_1) = \frac{1}{2}[2\bar{\sigma}(L, l_1) - \bar{\sigma}(L, 0) - \bar{\sigma}(L, L)]. \tag{2.16}$$

$\bar{\sigma}(L, 0)$ gives the specified dependence of stress on path length for monotonic decrease of strain from zero to $-E$. Provided that

$$\bar{\sigma}(L, 0) = -\bar{\sigma}(L, L), \tag{2.17}$$

we obtain from (2.16)

$$\sigma(L, l_1) = \bar{\sigma}(L, l_1). \tag{2.18}$$

The condition (2.17) is the condition that the prescribed stress merely changes sign if the prescribed strain history is replaced by its negative. That such a condition is necessary if the prescribed stress is to be expressible in the form (2.4) is evident. We thus conclude that if the stress at $l = L$, following a single reversal, changes sign when the strain history is replaced by its negative, then this stress can be expressed in the form (2.4).

We shall now show that if, further, the specified stress at $l = L$ following μ reversals of strain is an odd functiion of the strain history and satisfies the relations (2.10) for $\mu = 1, 2, \ldots, \mu$, then the specified stress can be expressed in the form (2.4) with $f(L, l)$ given by (2.15).

We prove this result by induction. Suppose that $\sigma(L, l_1, \ldots, l_{\alpha-1})$ with α odd is given by (2.4), i.e. by (2.6) with $\mu = \alpha - 1$, thus

$$\sigma = 2[g(L, l_1) - g(L, l_2) + \ldots - g(L, l_{\alpha-1})] + g(L, L). \tag{2.19}$$

We assume that (2.10) with $\mu = \alpha$ is valid. Then, we obtain, from (2.19), (2.9) and (2.10) with $\mu = \alpha$,

$$\sigma(L, l_1, \ldots, l_\alpha) = 2[g(L, l_1) - g(L, l_2) + \ldots + g(L, l_\alpha)] - g(L, L). \tag{2.20}$$

It follows that $\sigma(L, l_1, \ldots, l_\alpha)$ is given by (2.4) with $f(L, l)$ given by (2.15). An analogous argument is applicable to the case when α is even. Since we have already seen that the proposed theorem is valid when $\alpha = 1$, it is valid for all α.

It is suggested that an experimental test of the validity or non-validity of the constitutive eqn (2.4) might well be based on the relations (2.10).

3. THE INCREMENTAL MODULUS—UNIAXIAL DEFORMATION

In this section, we again consider a uniaxial deformation in which the stress is given in terms of the strain history, regarded as a function of strain path length l, by

$$\sigma(L) = \int_0^L f(L, l)\mathrm{d}\varepsilon(l), \tag{3.1}$$

where L is the value of l at which the stress is measured.

We consider a strain history which consists either of a monotonically increasing strain, or of a sequence of increasing and decreasing strains. We assume, however, that the strain E corresponding to path length L is positive and is reached finally through increasing strain. We calculate the incremental moduli at $l = L$ when the strain is increased by an infinitesimal amount ΔE. Let $\Delta\sigma$ be the corresponding increase in stress. If ΔL is the increase in l corresponding to the strain increase ΔE, we have

$$\Delta L = |\Delta E|. \tag{3.2}$$

We denote by μ_+ and μ_- the incremental moduli corresponding to ΔE positive and negative respectively. Then, from (3.1),

$$\begin{aligned} \mu_+ &= \frac{\Delta\sigma}{\Delta E} = \frac{\Delta\sigma}{\Delta L} = f(L, L) + \int_0^L \frac{\partial}{\partial L} f(L, l)\mathrm{d}l, \\ \mu_- &= \frac{\Delta\sigma}{\Delta E} = -\frac{\Delta\sigma}{\Delta L} = f(L, L) - \int_0^L \frac{\partial}{\partial L} f(L, l)\mathrm{d}l. \end{aligned} \tag{3.3}$$

We note that when $L = 0$,

$$\mu_+ = \mu_- = f(0, 0) = \mu_0, \text{ say}. \tag{3.4}$$

μ_0 is the incremental modulus at zero deformation. From (3.3)

$$\mu_+ + \mu_- = 2f(L, L). \tag{3.5}$$

Valanis [1, 2] made the particular choice of $f(L, l)$:

$$f(L, l) = E_0 + E_1\left(\frac{1+\beta l}{1+\beta L}\right)^{n-1}, \tag{3.6}$$

where E_0, E_1, β and n are positive constants. In this case,

$$\mu_0 = f(L, L) = E_0 + E_1, \tag{3.7}$$

and (3.5) becomes

$$\mu_+ + \mu_- = 2\mu_0. \tag{3.8}$$

For metals it is found experimentally that μ_- is approximately μ_0 and since, in the plastic regime, μ_+ is very much less than μ_0, the relation (3.8) is not valid. This was adduced as an argument against the endochronic theory advanced by Valanis in [1, 2] and led the latter to modify his theory. It is evident that the same criticism would apply to any theory in which $f(L, l)$ has the form

$$f(L, l) = C_1\frac{\phi_1(l)}{\phi_1(L)} + C_2\frac{\phi_2(l)}{\phi_2(L)} + \ldots + C_\nu\frac{\phi_\nu(l)}{\phi_\nu(L)}, \tag{3.9}$$

where $C_1, \ldots, C_\nu$ are constants, so that $f(L, L) = f(0, 0)$. However, it may not apply for other forms of $f(L, l)$. Also, this criticism does not, of course, preclude the applicability of one-dimensional constitutive equations of the type considered to other materials than metals.

4. DISSIPATION IN CYCLIC DEFORMATION

(a) Unloading–loading

We now consider the material to undergo a monotonically increasing uniaxial strain to strain ε_1. Thereafter, the strain is decreased to ε_2 and then increased to ε_1. We shall calculate the dissipation $\mathscr{D}$, per unit volume, in this cycle of deformation.

Let l_1, l_2 and L denote the strain path lengths at the first and second strain reversals and at the final strain ε_1 respectively. Then,

$$l_1 = \varepsilon_1, \quad l_2 = 2\varepsilon_1 - \varepsilon_2, \quad L = 3\varepsilon_1 - 2\varepsilon_2. \tag{4.1}$$

Let l be the value of the strain path length at a generic point on the cycle at which the strain is ε. Then,

$$\mathrm{d}l = \mathrm{d}\varepsilon(\text{strain-increasing}) \text{ and } \mathrm{d}l = -\mathrm{d}\varepsilon(\text{strain-decreasing}). \tag{4.2}$$

From (2.4) and (4.2) the stress at a generic point of the cycle, corresponding to path length l, is given by

$$\sigma(l, l_1) = 2g(l, l_1) - g(l, l) \text{ strain-decreasing}$$
$$\sigma(l, l_1, l_2) = 2g(l, l_1) - 2g(l, l_2) + g(l, l), \text{strain-increasing}, \tag{4.3}$$

where $g(l, \xi)$ is defined by (see eqn. 2.7)

$$g(l, \xi) = \int_0^\xi f(l, \xi)\mathrm{d}\xi. \tag{4.4}$$

The dissipation $\mathscr{D}$, per unit volume, in the cycle is given by

$$\mathscr{D} = -\int_{l_1}^{l_2} \sigma(l, l_1)\mathrm{d}l + \int_{l_2}^{L} \sigma(l, l_1, l_2)\mathrm{d}l. \tag{4.5}$$

We now make the assumption that the amplitude of the deformation is sufficiently small so that terms of second degree in $(l - l_1)$ may be neglected in comparison with those of first degree. Then, we may write, from (4.4),

$$\begin{aligned} g(l, l_1) &= g(l_1, l_1) + (l - l_1)g_1, \\ g(l, l_2) &= g(l_1, l_1) + (l - l_1)g_1 + (l_2 - l_1)g_2, \\ g(l, l) &= g(l_1, l_1) + (l - l_1)g_1 + (l - l_1)g_2, \end{aligned} \tag{4.6}$$

where g_1 and g_2 are defined as the values of $\partial g(l, \xi)/\partial l$ and $\partial g(l, \xi)/\partial \xi$ respectively when $l = l_1$ and $\xi = l_1$. Introducing (4.6) into (4.3), we obtain

$$\begin{aligned} \sigma(l, l_1) &= g(l_1, l_1) + (l - l_1)g_1 - (l - l_1)g_2, \\ \sigma(l, l_1, l_2) &= g(l_1, l_1) + (l - l_1)g_1 + (l + l_1 - 2l_2)g_2. \end{aligned} \tag{4.7}$$

We now introduce (4.7) into (4.5) and carry out the integrations to obtain, with (4.1),

$$\mathscr{D} = g_1(\varepsilon_1 - \varepsilon_2)^2. \tag{4.8}$$

For the particular expression for $f(l, \xi)$ used by Valanis (see eqn 3.6), viz.

$$f(l, \xi) = E_0 + E_1\left(\frac{1 + \beta\xi}{1 + \beta l}\right)^{n-1}, \tag{4.9}$$

where E_0, E_1, β and n are positive constants, we have from (4.4) and the definition of g_1:

$$g_1 = -\frac{E_1(n-1)}{n}\left[1 - \frac{1}{(1 + \beta l_1)^n}\right]. \tag{4.10}$$

(b) Loading–unloading

We now consider the material to undergo a monotonically increasing uniaxial strain to ε_1. It then undergoes a cycle of strain in which the strain is increased to $\varepsilon_2 > \varepsilon_1$ and then decreased to ε_1 again. The strain path lengths l_1, l_2 and L are now given by

$$l_1 = \varepsilon_1, \; l_2 = \varepsilon_2, \; L = 2\varepsilon_2 - \varepsilon_1. \tag{4.11}$$

If, as before, l is the value of the strain path length at a generic point on the cycle at which the strain is ε, we have, as before, the relations (4.2). The stress is given by

$$\begin{aligned} \sigma(l, l) &= g(l, l) \quad \text{strain-increasing} \\ \sigma(l, l_2) &= 2g(l, l_2) - g(l, l) \quad \text{strain-decreasing}. \end{aligned} \tag{4.12}$$

The dissipation $\mathscr{D}$ in the cycle is now given by

$$\mathscr{D} = \int_{l_1}^{l_2} \sigma(l, l)\mathrm{d}l - \int_{l_2}^{L} \sigma(l, l_2)\mathrm{d}l. \tag{4.13}$$

Again, with the small amplitude assumption, we obtain

$$\begin{aligned} g(l, l) &= g(l_1, l_1) + (l - l_1)g_1 + (l - l_1)g_2, \\ g(l, l_2) &= g(l_1, l_1) + (l - l_1)g_1 + (l_2 - l_1)g_2. \end{aligned} \tag{4.14}$$

From (4.12), (4.13) and (4.14) we obtain, with (4.11),

$$\mathscr{D} = -g_1(\varepsilon_2 - \varepsilon_1)^2. \tag{4.15}$$

If $f(l, \xi)$ is given by (4.9), we find, as before, that g_1 is given by (4.10).

(*c*) *Discussion*

We note from (4.4) that

$$g_1 = \frac{\partial g(l, \xi)}{\partial l}\bigg|_{l, \xi = l_1, l_1} = \int_0^{l_1} \frac{\partial f(l, \xi)}{\partial l}\,\mathrm{d}\xi. \tag{4.16}$$

Then, $g_1 = 0$ if $l_1 = 0$ and, from (4.8) and (4.15), we have $\mathscr{D} = 0$, whether the infinitesimal cycle of deformation is the loading–unloading or the unloading–loading cycle. If f is a monotonically decreasing function of l, then g_1 is negative and, for $l_1 \neq 0$, $\mathscr{D} < 0$ for the unloading–loading cycle and $\mathscr{D} > 0$ for the loading–unloading cycle. This is the situation which prevails in the case when f takes the particular form (4.9) proposed by Valanis. These results imply that the materials considered have no finite elastic range as was pointed out by Lee [3]. Also, the fact that $\mathscr{D} < 0$ for the unloading–loading cycle implies, as was pointed out by Sandler [4], that the material modeled is unstable.

5. THREE-DIMENSIONAL THEORY

We consider the material to undergo sufficiently small deformations so that the deformation in an element can be described by the history of the infinitesimal strain matrix $\boldsymbol{\varepsilon}(t) = \| \varepsilon_{ij}(t) \|$ referred to a rectangular cartesian coordinate system X. Let $\boldsymbol{\sigma} = \| \sigma_{ij} \|$ denote the stress matrix at time T, referred to the system x.

The length of the strain path at time t in 9-dimensional strain space is given by†

$$\mathrm{d}l(t) = \{tr[\mathrm{d}\boldsymbol{\varepsilon}(t)]^2\}^{1/2}, \quad l(0) = 0, \tag{5.1}$$

i.e.

$$l(t) = \int_0^t \{tr[\mathrm{d}\boldsymbol{\varepsilon}(t)]^2\}^{1/2}. \tag{5.2}$$

We introduce the notation

$$L = l(t). \tag{5.3}$$

It has been shown by Pipkin and Rivlin [6] that, for an isotropic rate-independent material, the stress corresponding to strain path length L is an isotropic tensor functional of the strain history regarded as a function of l. We shall examine the properties of a particular class of such relations which includes that used by Valanis [1, 2] as a special case.

We define an isotropic scalar functional ζ of $\boldsymbol{\varepsilon}(l)$ by

$$\mathrm{d}\zeta = \mathrm{d}\zeta(l) = \phi[\mathrm{d}\boldsymbol{\varepsilon}(l)], \quad \zeta(0) = 0, \tag{5.4}$$

where ϕ is a positive definite isotropic scalar function of $\mathrm{d}\boldsymbol{\varepsilon}(l)$, homogeneous of degree unity in the latter. From (5.4)

$$\zeta = \zeta(l) = \int_0^l \phi[\mathrm{d}\boldsymbol{\varepsilon}(l')]. \tag{5.5}$$

†The strain path lengths l and L introduced in this section do not reduce in the uniaxial case to those introduced in Section 2. They can, however, be simply related to them.

We call ζ the *intrinsic* time. We note that since ϕ is positive definite, ζ increases monotonically with l. Since ϕ is an isotropic function of $d\varepsilon(l)$, it must be expressible as a function of the elements of an isotropic integrity basis for $d\varepsilon(l)$. This may be chosen as I_1, I_2, I_3 defined by

$$I_1 = tr[d\varepsilon(l)], \quad I_2 = tr[d\varepsilon(l)]^2, \quad I_3 = \det[d\varepsilon(l)]. \tag{5.6}$$

We adopt the notation

$$Z = \zeta(L). \tag{5.7}$$

We now assume that the stress σ corresponding to strain path length L is given by an expression of the form

$$\sigma = 2\int_0^L \mu(Z,\zeta)d\varepsilon(l) + \delta\int_0^L \lambda(Z,\zeta)tr[d\varepsilon(l)], \tag{5.8}$$

where δ denotes the unit matrix.

Valanis adopts the particular form for ϕ:

$$\phi[d\varepsilon(l)] = (k_1 I_1^2 + k_2 I_2)^{1/2}, \tag{5.9}$$

where k_1 and k_2 are positive constants, which may vary from material to material. He gives no reason for making this particular choice beyond the statement "it appears logical to define ζ by...".

6. SIMULTANEOUS SIMPLE EXTENSION AND SIMPLE SHEAR

We now suppose that the material is subjected to a simple extension in the x_1-direction of a rectangular cartesian coordinate system x and a simple shear for which the direction of shear is the x_2-direction and the plane of shear is the x_1x_2-plane. We adopt the notation

$$\varepsilon = \varepsilon(l) = \varepsilon_{11}(l), \quad \kappa = \kappa(l) = \varepsilon_{12}(l) \tag{6.1}$$

and

$$E = \varepsilon(L), K = \kappa(L), \tag{6.2}$$

where $l = L$ at the instant at which the stress is measured.

In order to be specific we shall suppose that the initial extension is positive and that successive reversals of the extension occur at $\varepsilon = \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\mu$. We also assume that the initial shear is positive and successive reversals occur at $K = \kappa_1, \kappa_2, \ldots, \kappa_\nu$. It is assumed that the shearing and extensional deformations may occur in any order, but take place in disjoint time intervals: Let

$$\begin{aligned} l &= l_\alpha \text{ when } \varepsilon = \varepsilon_\alpha (\alpha = 1, \ldots, \mu), \\ l &= \bar{l}_\beta \text{ when } \kappa = \kappa_\beta (\beta = 1, \ldots, \nu). \end{aligned} \tag{6.3}$$

For a simple extension in the x_1-direction, only σ_{11} is non-zero and, from (5.8), for the simple shearing deformation in the 12-plane, only $\sigma_{12}(=\sigma_{21})$ is non-zero.

We assume, with Valanis, that Poisson's ratio for the material is a constant $\tilde{\omega}$, say, independent of the deformation to which the material is subjected. (We shall discuss this restriction later in Section 7.) Then,

$$tr\,\varepsilon = (1 - 2\tilde{\omega})\varepsilon. \tag{6.4}$$

We have, from (5.2), (6.3) and the assumption that Poisson's ratio is constant

$$l = (1+2\bar{\omega}^2)^{1/2}\left\{\sum_{\alpha=1}^{\beta} 2(-1)^{\alpha-1}\,\varepsilon_\alpha + (-1)^\alpha\,\varepsilon\right\} + 2^{1/2}\left\{\sum_{\alpha=1}^{\gamma} 2(-1)^{\alpha-1}\,\kappa_\alpha + (-1)^\alpha\,\kappa\right\}, \tag{6.5}$$

where β is the number of reversals of extensional strain and γ is the number of reversals of shear strain prior to the strain-path length l being attained.

Since ϕ is homogeneous of degree unity in $\mathrm{d}\varepsilon(l)$ and is positive definite, for the particular class of deformations considered in this section

$$\phi = \begin{matrix} \bar{k}_1\,|\mathrm{d}\varepsilon(l)| & \text{for simple extension} \\ \bar{k}_2\,|\mathrm{d}\kappa(l)| & \text{for simple shear,} \end{matrix} \tag{6.6}$$

where $\bar{k}_1$ and $\bar{k}_2$ are positive constants, which depend on the particular choice of ϕ. We note from (6.5) that

$$\begin{aligned} |\mathrm{d}\varepsilon(l)| &= (1+2\bar{\omega}^2)^{-1/2}\,\mathrm{d}l \\ |\mathrm{d}\kappa(l)| &= 2^{-1/2}\,\mathrm{d}l. \end{aligned} \tag{6.7}$$

From (6.4) and (5.6), we have

$$\begin{matrix} I_1 = (1-2\bar{\omega})\mathrm{d}\varepsilon,\ \ I_2 = (1+2\bar{\omega}^2)(\mathrm{d}\varepsilon)^2,\ \ I_3 = \bar{\omega}^2(\mathrm{d}\varepsilon)^3 & \text{for simple extension} \\ I_1 = I_3 = 0,\ \ I_2 = 2(\mathrm{d}\kappa)^2 & \text{for simple shear.} \end{matrix} \tag{6.8}$$

In the particular case (5.9) considered by Valanis, we obtain, with (6.7) and (6.8),

$$\begin{matrix} \phi = \{k_1(1-2\bar{\omega})^2 + k_2(1+2\bar{\omega}^2)\}^{1/2}(1+2\bar{\omega}^2)^{-1/2}\,\mathrm{d}l & \text{for simple extension} \\ \phi = k_2^{1/2}\,\mathrm{d}l & \text{for simple shear.} \end{matrix} \tag{6.9}$$

Introducing (6.7) into (6.6) and comparing the resulting expressions with (6.9), we obtain

$$\begin{aligned} \bar{k}_1 &= \{k_1(1-2\bar{\omega})^2 + k_2(1+2\bar{\omega}^2)\}^{1/2}, \\ \bar{k}_2 &= (2k_2)^{1/2}. \end{aligned} \tag{6.10}$$

As another illustration, suppose

$$\phi = [aI_1^4 + bI_1^2I_2 + cI_3I_1 + eI_2^2]^{1/4}. \tag{6.11}$$

where a,b,c,e are positive constants and I_1, I_2, I_3 are defined by (5.6). Then, from (6.11), (6.7) and (6.8)

$$\begin{aligned} \phi &= [a(1-2\bar{\omega})^4 + b(1-2\bar{\omega})^2(1+2\bar{\omega}^2) + c\bar{\omega}^2(1-2\bar{\omega}) \\ &\quad + e(1+2\bar{\omega}^2)^2]^{1/4}(1+2\bar{\omega}^2)^{-1/2}\,\mathrm{d}l \quad \text{for simple extension} \\ \phi &= e^{1/4}\,\mathrm{d}l \quad \text{for simple shear.} \end{aligned} \tag{6.12}$$

Comparing (6.12) with the expressions for ϕ obtained from (6.6) and (6.7), we have

$$\begin{aligned} \bar{k}_1 &= [a(1-2\bar{\omega})^4 + b(1-2\bar{\omega})^2(1+2\bar{\omega}^2) + c\bar{\omega}^2(1-2\bar{\omega}) + e(1+2\bar{\omega}^2)^2]^{1/4}, \\ \bar{k}_2 &= (4e)^{1/4}. \end{aligned} \tag{6.13}$$

We have seen that any ϕ, which is a function of I_1, I_2, I_3 and is homogeneous of degree 1 in $d\varepsilon/dl$, leads to the expression (6.6) for simple extensions and simple shears of the types considered. It follows that we connot distinguish between different forms of ϕ by experiments in which the deformation is a sequence of such simple extensions and simple shears. In particular, the experiments of Mair and Pugh, which involve the superposition of simple extensions on simple torsions of thin tubes, cannot be used, as Valanis has done, to establish the validity of the particular form (5.9) for ϕ. Agreement of his theory with such experiments could, at best, lead only to the conclusion that the theory is not inconsistent with the experiments.

7. POISSON'S RATIO

It is assumed by Valanis in [1, 2] that Poisson's ratio for the materials he considers is constant. An assumption as radical as this should be tested independently for any material to which the theory is applied. This could be done, at any rate in principle, by measuring the change in volume, or the lateral contraction, of a rod of the material when subjected to simple extension.

Alternatively, it might be done by making simultaneous measurements of tensile and shearing force when a thin cylinder of the material is simultaneously subjected to simple extensional and shear strains which are increased proportionately. For simultanious monotonically increasing simple extensions and simple shears, we find from (5.8) that, whether or not Poisson's ratio $\bar{\omega}$ is constant

$$\begin{aligned}\sigma_{11} &= \int_0^L [2\mu(Z,\zeta) + (1-2\bar{\omega})\lambda(Z,\zeta)]\,d\varepsilon(l),\\ \sigma_{22} = \sigma_{33} &= \int_0^L [-2\bar{\omega}\mu(Z,\zeta) + (1-2\bar{\omega})\lambda(Z,\zeta)]\,d\varepsilon(l) = 0,\\ \sigma_{12} &= \int_0^L 2\mu(Z,\zeta)\,d\kappa(l),\quad \sigma_{23} = \sigma_{31} = 0,\end{aligned} \tag{7.1}$$

where $\varepsilon(l)$ and $\kappa(l)$ denote the extensional and shear strains respectively. In (7.1), l is given, from (5.1), by

$$dl = [(1+2\bar{\omega}^2)(d\varepsilon)^2 + 2(d\kappa)^2]^{1/2}, \tag{7.2}$$

with $l = 0$ when the material is undeformed. Also, from (5.6), we have

$$\begin{aligned}I_1 = (1-2\bar{\omega})d\varepsilon(l),\ I_2 &= (1+2\bar{\omega}^2)[d\varepsilon(l)]^2 + 2[d\kappa(l)]^2,\\ I_3 &= \bar{\omega}[d\varepsilon(l)]\{\bar{\omega}[d\varepsilon(l)]^2 + [d\kappa(l)]^2\}.\end{aligned} \tag{7.3}$$

$d\zeta$ is a positive function of I_1, I_2, I_3, homogeneous of degree unity in $d\varepsilon(l)$ and $d\kappa(l)$, and $\zeta = 0$ when $l = 0$.

It follows from (7.1) that, if $\bar{\omega}$ is constant,

$$\begin{aligned}\sigma_{11} &= 2(1+\bar{\omega})\int_0^L \mu(Z,\zeta)\,d\varepsilon(l),\\ \sigma_{12} &= 2\int_0^L \mu(Z,\zeta)\,d\kappa(l).\end{aligned} \tag{7.4}$$

Now, if the extensional and shear strains are increased proportionately, so that

$$\kappa(l) = \chi\varepsilon(l), \tag{7.5}$$

where χ is constant, we have, from (7.2) and (7.5),

$$\begin{aligned} d\varepsilon(l) &= (1+2\bar{\omega}^2+2\chi^2)^{-1/2}\,dl,\\ d\kappa(l) &= \chi(1+2\bar{\omega}^2+2\chi^2)^{-1/2}\,dl. \end{aligned} \tag{7.6}$$

Also, from (7.4) and (7.6), we have

$$\begin{aligned} \sigma_{11} &= 2(1+\bar{\omega})(1+2\bar{\omega}^2+2\chi^2)^{-1/2}\int_0^L \mu(Z,\zeta)dl,\\ \sigma_{12} &= 2(1+2\bar{\omega}^2+2\chi^2)^{-1/2}\chi\int_0^L \mu(Z,\zeta)dl. \end{aligned} \tag{7.7}$$

Thus,

$$\sigma_{11}/\sigma_{12} = (1+\bar{\omega})/\chi. \tag{7.8}$$

The assumption that Poisson's ratio is constant has been criticized by Lee [3] insofar as the applicability of the theory to metals is concerned. For most metals, Poisson's ratio in the elastic range is about 1/4. This implies that the ratio bulk modulus/Young's modulus $\approx 2/3$. It is well-known that when the plastic strain in a metal is large compared with the elastic strain, the bulk modulus is far greater than the Young's modulus.†

It is, however, possible that for other materials, particularly those which may, with good approximation, be regarded as incompressible, the assumption of constant Poisson's ratio will be valid.

8. A CONTINUITY CONSIDERATION

We now consider a deformation which consists of alternate infinitesimal simple extensional strains of amount $\Delta\varepsilon$ and infinitesimal simple shearing strains of amount $\Delta\kappa$. As in Section 6, we consider the extension to be parallel to the x_1-axis of a rectangular cartesian coordinate system x and the shear to be in the x_2-direction and the x_1x_2-plane. We consider the tensile and shearing stresses after ν such simple extensions and simple shears. We denote the resultant extensional strain by $E(=\nu\Delta\varepsilon)$ and the resultant shear strain by $K(=\nu\Delta\kappa)$. The value of ζ when the extensional strain is ε and the shearing strain is κ is given by (6.6)

$$\zeta = \bar{k}_1\varepsilon + \bar{k}_2\kappa. \tag{8.1}$$

Since

$$\kappa/\varepsilon = K/E, \tag{8.2}$$

eqn (8.1) can be rewritten at

$$\zeta = \left(\bar{k}_1 + \frac{K}{E}\bar{k}_2\right)\varepsilon = \left(\frac{E}{K}\bar{k}_1 + \bar{k}_2\right)\kappa. \tag{8.3}$$

We shall assume that ζ has the form proposed by Valanis and given by (5.5) and (5.9). Then, $\bar{k}_1$ and $\bar{k}_2$ are given by (6.10).

With the assumption that Poisson's ratio is constant, the tensile and shear stresses are obtained from (7.4) as

$$\begin{aligned} \sigma_{11} &= 2(1+\bar{\omega})\int_0^E \mu(Z,\zeta)d\varepsilon,\\ \sigma_{12} &= 2\int_0^K \mu(Z,\zeta)d\kappa, \end{aligned} \tag{8.4}$$

†Here we are using the term "Young's modulus" in the sense of (tensile force per unit area/extensional strain) notwithstanding that the material is plastically deformed.

where ζ is given by (8.3) and

$$Z = \bar{k}_1 E + \bar{k}_2 K. \tag{8.5}$$

We now suppose that instead of increasing the strains to their final values in a stepwise fashion we increase them proportionately. Then, using the expression for ζ given by (5.5), (7.3) and (5.9) we have, with (8.2) and (6.10),

$$\zeta = \left(\bar{k}_1^2 + \frac{K^2}{E^2}\bar{k}_2^2\right)^{1/2} \varepsilon = \left(\bar{k}_1^2 \frac{E^2}{K^2} + \bar{k}_2^2\right)^{1/2} \kappa. \tag{8.6}$$

Z is obtained by taking $\varepsilon = E$ or $\kappa = K$ in (8.6).

The tensile and shear stresses are then given by (8.4) with these expressions for ζ and Z. It is evident that they are, in general, different from those which correspond to the stepwise deformation previously considered. Similar disparities will evidently be found, except perhaps in some exceptional cases, if other forms for ζ are adopted.

In mathematical terms, the type of behavior predicted by the model we have considered arises from the fact that the intrinsic time is not a continuous functional, in the sense of the supremum norm, of the strain history.

We note that if a material does, in fact, exhibit the type of behavior predicted in this section, it will be extremely difficult, and perhaps impossible, to subject it to meaningful tests of the type considered.

9. VALANIS'S SECOND THEORY—PHENOMENOLOGICAL APPROACH

In presenting his first theory, Valanis claimed, as a major advance, that the endochronic assumption enables us to construct a continuum-mechanical theory, for rate-independent elastic–plastic materials, in which the material properties are described by a single constitutive equation without the need to introduce the concept of a yield surface. These constitutive equations also have the attractive feature that, in them, the stress is related to *total* strain.

Both of these features are lost in the theory, advanced by Valanis in order to meet some of the criticisms of his first theory. This second theory is motivated by "thermodynamic" arguments of a type similar to and open to the same criticisms as, those advanced in developing the first theory.

In the present section we will discuss the theory in more conventional phenomenological terms. In [5] Valanis presents his ideas at various levels of generality, but here we will limit our discussion to that form of the theory whose implications are discussed in [5] at greatest length.

The strain $\boldsymbol{\varepsilon}$ is regarded as the sum of an elastic strain $\boldsymbol{\varepsilon}_E$ and a plastic strain $\boldsymbol{\varepsilon}_P$, thus:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_E + \boldsymbol{\varepsilon}_P. \tag{9.1}$$

The stress $\boldsymbol{\sigma}$ is related to the elastic strain by a constitutive relation of the form

$$\boldsymbol{\sigma} = 2\mu_0 \boldsymbol{\varepsilon}_E + \lambda_0 (tr\boldsymbol{\varepsilon}_E)\boldsymbol{\delta}, \tag{9.2}$$

where μ_0 and λ_0 are constants. It is assumed that dilatational deformations are purely elastic, so that

$$tr\,\boldsymbol{\varepsilon}_P = 0, \quad tr\,\boldsymbol{\varepsilon} = tr\,\boldsymbol{\varepsilon}_E. \tag{9.3}$$

From (9.2) and $(9.3)_2$, we have

$$tr\,\boldsymbol{\sigma} = (2\mu_0 + 3\lambda_0) tr\,\boldsymbol{\varepsilon}. \tag{9.4}$$

From (9.2), the deviatoric stress $\mathbf{s}$ is given by

$$\mathbf{s} = \boldsymbol{\sigma} - \frac{1}{3}(tr\,\boldsymbol{\sigma}) = 2\mu_0 \mathbf{e}_E, \tag{9.5}$$

where $\mathbf{e}_E$ denotes the deviatoric elastic strain. From (9.1), (9.3) and (9.5), the deviatoric plastic strain $\mathbf{e}_P$ is given by

$$\mathbf{e}_P = \boldsymbol{\varepsilon}_P = \mathbf{e} - \mathbf{e}_E = \mathbf{e} - \mathbf{s}/2\mu_0, \tag{9.6}$$

where e denotes the deviatoric total strain.

An intrinsic time ζ is defined in terms of the plastic strain by

$$\mathrm{d}\zeta = \{tr(\mathrm{d}\boldsymbol{\varepsilon}_P)^2\}^{1/2}, \tag{9.7}$$

with $\zeta = 0$ in the undeformed state. We note that ζ is the arc length measured along the strain path in the 9-dimensional space defined by the components of the plastic strain in a rectangular cartesian coordinate system. [We note the analogy between ζ, so defined, and l defined by (5.1).]

For a rate-independent material the plastic strain $\boldsymbol{\varepsilon}_P$ may be regarded as a function of ζ. It is assumed in [5] that the deviatoric stress, when $\zeta = Z$, is related to the plastic strain by a constitutive equation of the form

$$\mathbf{s} = \mathbf{s}(Z) = \mathbf{s}_Y^{(0)} \frac{\mathrm{d}\boldsymbol{\varepsilon}_P}{\mathrm{d}\zeta}\bigg|_{\zeta = Z} \psi(Z) + \mathbf{r}, \tag{9.8}$$

where

$$\mathbf{r} = 2\mu_0 \int_0^Z f(Z, \zeta)\mathrm{d}\boldsymbol{\varepsilon}_P(\zeta), \tag{9.9}$$

$\mathbf{s}_Y^{(0)}$ is a positive constant, and $\psi(Z)$ is a positive monotonically increasing function of Z. From (9.8) and (9.7) it follows that

$$tr(s - r)^2 = [s_Y^{(0)}\psi(Z)]^2. \tag{9.10}$$

Equation (9.10) may be regarded as a hypersphere in the 9-dimensional space formed by the components of s in a rectangular cartesian coordinate system. The center of this hypersphere is at r (regarded as a vector in the 9-dimensional space) and its radius is $\mathbf{s}_Y^{(0)}\psi(Z)$. Since $tr\, s = 0$, s must be on the intersection of the hyperplane $tr\, \mathbf{s} = 0$ with this hypersphere. This intersection is itself a hypersphere in the 8-dimensional sub-space of the 9-dimenional space for which $tr\, \mathbf{s} = 0$. This hypersphere is called the *yield surface*. Since $tr\, \boldsymbol{\varepsilon}_P = 0$, it follows that the point r lies in this subspace. It is, of course, the center of the hypersphere in the 8-dimensional space and the radius of this hypersphere is $\mathbf{s}_Y^{(0)}\psi(Z)$.

We note from (9.7) that if $\mathrm{d}\boldsymbol{\varepsilon}_P = 0$, then $\mathrm{d}\zeta = 0$ and, conversely, if $\mathrm{d}\zeta = 0$, then $\mathrm{d}\boldsymbol{\varepsilon}_P = 0$. Thus, if at any instant the deformation is purely elastic, $\mathrm{d}\boldsymbol{\varepsilon}_P/\mathrm{d}\zeta|_{\zeta=Z}$ is indeterminate and eqn (9.8) becomes meaningless. However, from (9.2), the stress increment $\mathrm{d}\boldsymbol{\sigma}$ is then related to the strain increment $\mathrm{d}\boldsymbol{\varepsilon}(=\mathrm{d}\boldsymbol{\varepsilon}_E)$ by

$$\mathrm{d}\boldsymbol{\sigma} = 2\mu_0\, \mathrm{d}\boldsymbol{\varepsilon} + \lambda_0(tr\, \mathrm{d}\boldsymbol{\varepsilon})\boldsymbol{\delta} \tag{9.11}$$

and, from (9.5), the deviatoric stress increment $\mathrm{d}s$ is related to the deviatoric strain increment $\mathrm{d}\boldsymbol{e}(=\mathrm{d}\boldsymbol{e}_E)$ by

$$\mathrm{d}s = 2\mu_0\, \mathrm{d}\boldsymbol{e}. \tag{9.12}$$

If $\mathrm{d}\boldsymbol{\varepsilon}_P/\mathrm{d}\zeta\,|_{\zeta=Z}$ is not indeterminate, it follows from (9.8) that the plastic strain path, at arc length Z, is in the direction of the outward normal to the yield surface corresponding to arc length Z.

We see that the theory which has been presented is of a type generally similar to many other plasticity theories which have been formulated. In effect, it is assumed that:

(i) The total strain may be regarded as the sum of an elastic strain and a plastic strain; the plastic strain is isochoric.

(ii) It is assumed that the moduli associated with changes of elastic strain are constants, independent of strain history.

(iii) There exists a spherical yield surface in deviatoric stress space, whose outward normal is parallel to the incremental plastic strain vector.

(iv) The radius of the yield surface and the position of its center depend on the history of the plastic strain (i.e. the material modeled exhibits both kinematic and isotropic hardening). A particular form is chosen for this dependence, which, while having a measure of generality, is far from being of the most general form that can be envisaged.

(v) If the deviatoric stress lies at a point on the yield surface corresponding to some specified plastic strain history and is then changed to a point lying inside, or on, this yield surface, the corresponding deformation is purely elastic.

It is evident that the assumptions (i), (iii) and (iv) meet the objections to the first theory of Valanis discussed in Sections 4, 5 and 7. Paralleling the discussion in Section 6, we shall now discuss the extent to which experiments involving superposed simple extensions and shears carried out in discrete time intervals can be used to establish the validity of the constitutive eqn (9.8), with (9.9).

We again consider a somewhat wider class of constitutive equations which have the form given in (9.8) and (9.9), but in which ζ is defined in a different manner. Let l_P denote the length of the plastic strain path at time t, thus (see eqn 9.7)

$$l_P = l_P(t) = \int_0^t \{tr[d\varepsilon_P(t)]^2\}^{1/2}, \tag{9.13}$$

and let

$$L_P = l_P(T). \tag{9.14}$$

Then, l_P can be used to parametrize ε_P.

We now define $d\zeta$ as an arbitrary positive isotropic scalar function of $d\varepsilon_P$, which is homogeneous of degree unity in the latter. Accordingly, $d\zeta$ is a positive function of the isotropic invariants of $d\varepsilon_P$, denoted I_1, I_2, I_3 and defined (see eqn 5.6)

$$I_1 = tr[d\varepsilon_P(l_P)] = 0, I_2 = tr[d\varepsilon_P(l_P)]^2, I_3 = \det[d\varepsilon_P(l_P)], \tag{9.15}$$

which is homogeneous of degree unity in $d\varepsilon_P$. We take $\zeta = 0$, when $l_P = 0$. We define Z by

$$Z = \zeta(L_P). \tag{9.16}$$

With this new definition of ζ, we assume a constitutive equation of the form (9.8), with (9.9). Then, it is easily shown, in a manner analogous to that employed in Section 6 in discussing the first theory, that measurements of stress, in experiments involving simple extensions and shears carried out in disjoint time intervals, cannot be used to establish the validity of any particular form of dependence of ζ on I_2 and I_3.

Also, it can be shown, paralleling the discussion in Section 8, that, in general, for a constitutive equation of the form given in (9.8) and (9.9), paths in plastic strain space which are arbitrarily close together, may have very different values for ζ corresponding to the same values of plastic strain and accordingly may yield very different values for the stress at the same value of the strain. However, this fact does not constitute an objection to the theory, since plastic strain histories which are close together, in the sense of the supremum norm, may be associated with very different histories of the total strain.

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APPENDIX. "THERMODYNAMIC" JUSTIFICATION OF THE FIRST THEORY

In this section, we outline the essential features of the argument of Valanis in arriving at the constitutive equations of his first theory of plasticity, in the particular case when the strains are small and the deformations are carried out isothermally. The argument given here is not strictly that given by Valanis[1]. His theory involves rather elaborate thermodynamic considerations of questionable validity, which are, in any case, unnecessary for attaining his final constitutive equations. We have found it possible to replace these arguments by far simpler ones which involve only some of the assumptions used, either explicitly or implicitly, by Valanis and none that are not used by him.

Valanis defines, for a rate-independent material, an *intrinsic time scale* z which is a monotonically non-decreasing function of real time t and is a functional of the strain-history up to time t. He considers that the form of this functional dependence may depend on the material considered. For an isotropic material undergoing small deformations, he expresses z as a monotonically increasing function of another (positive) variable ζ, thus

$$z = f(\zeta). \tag{A.1}$$

ζ is, in turn, defined by an expression of the form

$$(d\zeta)^2 = k_1(tr\, d\varepsilon)^2 + k_2 tr(d\varepsilon)^2, \tag{A.2}$$

where k_1 and k_2 are positive material constants, with $\zeta = 0$ initially. We note that the expression for $d\zeta$ is the most general quadratic form in $d\varepsilon$, with constant coefficients, which is invariant under an arbitrary orthogonal transformation. Valanis then assumes that the stress σ measured at time t is an isotropic tensor functional of the strain regarded as a function of z.

These assumptions ensure that if we consider the strain in the material to execute a specified path in nine-dimensional strain space, the stress corresponding to a particular point on this path is independent of the rate at which the path is executed. The same objective can be achieved by choosing z to be any isotropic scalar functional of the strain history which increases monotonically with time. For example, the strain path length l, defined by (5.2), provides such a functional. It is emphasized that the particular choice which is made neither increases nor decreases the generality of the theory developed. It merely changes the *form* of the functional expression for σ appropriate to a given material.

The particular definition (A.2) of ζ adopted by Valanis is justified only by the statement [1] "it appears logical to define ζ by...". The particular dependence of z on ζ which he assumes is avowedly justified only by alleged agreement of the predictions of the theory with experiment.

From this point onwards, the argument of Valanis can be very much simplified without losing his final result. We shall present this simpler argument here and avoid the questionable thermodynamic argument given by him in [1].

We assume, with Valanis, that the state of the material at any instant can be characterized by the instantaneous values of the strain ε and of ν internal variables $q^{(\alpha)}(\alpha = 1, \dots, \nu)$, which are symmetric second-order tensors It is further assumed, still following Valanis, that the rate of change of $q^{(\alpha)}$, measured with respect to z, is determined by the instantaneous values of ε and of $q^{(\beta)}(\beta = 1, \dots, \nu)$, thus

$$\frac{dq^{(\alpha)}}{dz} = f^{(\alpha)}(\varepsilon, q^{(\beta)}). \tag{A.3}$$

This is the evolution equation for $q^{(\alpha)}$. (We will see later that this assumption is unacceptable, but for the moment will pretend that this is not the case.)

The function $f^{(\alpha)}$ must be an *isotropic* symmetric tensor function of ε and $q^{(\beta)}$, since the material is isotropic. We shall suppose that it depends sufficiently smoothly on ε and $q^{(\beta)}$ and that these are sufficiently small, so that we can neglect terms of higher degree than the first in them. We also make the assumption that $\varepsilon = q^{(\alpha)} = 0$ when $z = 0$. We accordingly write

$$\frac{dq^{(\alpha)}}{dz} = A_\alpha \varepsilon + \sum_{\beta=1}^{\nu} A_{\alpha\beta} q^{(\beta)} + (B_\alpha tr\varepsilon + \sum_{\beta=1}^{\nu} B_{\alpha\beta} tr\, q^{(\beta)})\delta, \tag{A.4}$$

where the A's and B's sre constants and δ is the Kronecker delta. From (A.4) we readily obtain

$$\frac{d(tr\, q^{(\alpha)})}{dz} = (A_\alpha + 3B_\alpha) tr\, \varepsilon + \sum_{\beta=1}^{\nu} (A_{\alpha\beta} + 3B_{\alpha\beta}) tr\, q^{(\beta)},$$
$$\frac{dq_D^{(\alpha)}}{dz} = A_\alpha \varepsilon_D + \sum_{\beta=1}^{\nu} A_{\alpha\beta} q_D^{(\beta)}, \tag{A.5}$$

where the deviatoric strain and deviatoric internal variables are given by

$$\varepsilon_D = \varepsilon - \frac{1}{3}(tr\, \varepsilon)\delta, \; q_D^{(\alpha)} = q^{(\alpha)} - \frac{1}{3}(tr\, q^{(\alpha)})\delta. \tag{A.6}$$

Equations (A.5)$_2$ can be solved for $q_D^{(\alpha)}$ to yield a solution of the form

$$q_D^{(\alpha)} = \sum_{\beta=1}^{\nu} \bar{A}_{\alpha\beta} \int_0^z e^{-\rho_\beta(z-z')} \varepsilon_D(z')dz', \tag{A.7}$$

where the $\bar{A}$'s are constants determined by the equations

$$\sum_{\beta=1}^{\nu} \bar{A}_{\beta\mu}(A_{\alpha\beta} + \rho_\mu \delta_{\alpha\beta}) = 0, \quad (\alpha, \mu = 1, \dots, \nu)$$

$$\sum_{\beta=1}^{\nu} \bar{A}_{\alpha\beta} = A_\alpha, \tag{A.8}$$

in which $\rho_\mu(\mu = 1, \dots, \nu)$ are the solutions for ρ of the equation

$$\det|\rho\delta_{\alpha\beta} + A_{\alpha\beta}| = 0, \tag{A.9}$$

where $\delta_{\alpha\beta}$ is the ν-dimensional Kronecker delta.

In a similar manner eqns $(A.5)_1$ can be solved for $tr\, q^{(\alpha)}$ to yield

$$tr\, q^{(\alpha)} = \sum_{\beta=1}^{\nu} \bar{B}_{\alpha\beta} \int_0^z e^{-\lambda_\beta(z-z')} tr\, \varepsilon(z')\mathrm{d}z', \tag{A.10}$$

where the B's are constants given by

$$\sum_{\beta=1}^{\nu} \bar{B}_{\beta\mu}(A_{\alpha\beta} + 3B_{\alpha\beta} + \lambda_\mu \delta_{\alpha\beta}) = 0, \quad (\alpha, \mu = 1, \dots, \nu)$$

$$\sum_{\beta=1}^{\nu} \bar{B}_{\alpha\beta} = A_\alpha + 3B_\alpha, \tag{A.11}$$

in which $\lambda_\mu(\mu = 1, \dots, \nu)$ are the solutions for λ of the equation

$$\det|\lambda\delta_{\alpha\beta} + A_{\alpha\beta} + 3B_{\alpha\beta}| = 0. \tag{A.12}$$

In deriving (A.7) and (A.10), we make the assumption that $q^{(\alpha)} = 0$ when $z = 0$.

With the assumption that $\varepsilon(z) = 0$ when $z = 0$, we can require (A.7) and (A.10) as

$$q_D^{(\alpha)} = \sum_{\beta=1}^{\nu} \bar{A}_{\alpha\beta}\left\{\frac{1}{\rho_\beta}\varepsilon_D(z) - \frac{1}{\rho_\beta}\int_0^z e^{-\rho_\beta(z-z')}\mathrm{d}\varepsilon_D(z')\right\},$$

$$tr\, q^{(\alpha)} = \sum_{\beta=1}^{\nu} \bar{B}_{\alpha\beta}\left\{\frac{1}{\lambda_\beta} tr\, \varepsilon(z) - \frac{1}{\lambda_\beta}\int_0^z e^{-\lambda_\beta(z-z')}\mathrm{d}[tr\, \varepsilon(z')]\right\}. \tag{A.13}$$

We now assume that the Cauchy stress $\boldsymbol{\sigma}$ at "time" z is an isotropic linear function of $\varepsilon(z)$ and $q^{(\alpha)}(\alpha = 1, \dots, \nu)$. The deviatoric stress $\boldsymbol{\sigma}_D$ is defined by

$$\boldsymbol{\sigma}_D = \boldsymbol{\sigma} - \frac{1}{3}(tr\, \boldsymbol{\sigma})\boldsymbol{\delta}. \tag{A.14}$$

Then, we may express $\boldsymbol{\sigma}_D$ and $tr\, \boldsymbol{\sigma}$ in the forms

$$\boldsymbol{\sigma}_D = G\varepsilon_D + \sum_{\beta=1}^{\nu} G_\beta q_D^{(\beta)},$$

$$tr\, \boldsymbol{\sigma} = H(tr\, \varepsilon) + \sum_{\beta=1}^{\nu} H_\beta(tr\, q^{(\beta)}), \tag{A.15}$$

where the G's and H's are constants. Using (A.13), we obtain from (A.15), expressions for $\boldsymbol{\sigma}_D$ and $tr\, \boldsymbol{\sigma}$ of the forms

$$\boldsymbol{\sigma}_D = 2\int_0^z \mu(z - z')\mathrm{d}\varepsilon_D(z'),$$

$$tr\boldsymbol{\sigma} = \int_0^z \kappa(z - z')\mathrm{d}[tr\varepsilon], \tag{A.16}$$

where the functions $\mu(z - z')$ and $\kappa(z - z')$ have the forms

$$\mu(z - z') = \mu_0 + \sum_{\beta=1}^{\nu} \mu_\beta\, e^{-\rho_\beta(z-z')},$$

$$\kappa(z - z') = \kappa_0 + \sum_{\beta=1}^{\nu} \kappa_\beta\, e^{-\lambda_\beta(z-z')}, \tag{A.17}$$

and μ_β, κ_β $(\beta = 0, \dots, \nu)$ are constants.

It is evident that the passage from eqns (A.13) and (A.15) to constitutive eqns (A.16) is valid, whatever the physical interpretation of z, provided that it is an isotropic variable in terms of which the histories of the strain and of the internal variables can be parametrized. The choice of z made by Valanis is

$$z = f(\zeta), \quad f(\zeta) = \frac{1}{\beta}\ln(1 + \beta\zeta), \tag{A.18}$$

where β is a positive constant and ζ is defined by (A.2).

If we accept a relation of the form (A.18)$_1$ and use (A.3), we obtain

$$\frac{dq^{(\alpha)}}{d\zeta} = f'(\zeta)f^{(\alpha)}(\varepsilon, q^{(\beta)}). \tag{A.19}$$

This implies that the infinitesimal changes in the internal variables due to an infinitesimal change in the strain depends not only on the instantaneous values of the internal variables and the strain, but also, through ζ, on the whole past history of the strain. We remark that usually, when internal variables are introduced, along with the current strain, as independent variables in a constitutive equation, it is in order to provide a full descritption of the current state in terms of the current values of the independent variables of the theory. If the internal variables and the strain provide a complete description of the state, then the infinitesimal change in the internal variables due to a specified infinitesimal change in the strain should depend only on their current values.

Another peculiarity of the relation (A.19) and, indeed, of (A.3) is that different infinitesimal changes of strain which result, from (A.2), in the same values of $d\zeta$, lead to the same infinitesimal changes in the internal variables $q^{(\alpha)}$. This peculiarity can be avoided by including the "rate" of change of strain, $d\varepsilon/dz$, as an independent variable in (A.3). Then, the relations (A.5) are replaced by

$$\frac{d(trq^{(\alpha)})}{dz} = (A_\alpha + 3B_\alpha)tr\varepsilon + (C_\alpha + 3D_\alpha)\frac{d(tr\varepsilon)}{dz} + \sum_{\beta=1}^{\nu}(A_{\alpha\beta} + 3B_{\alpha\beta})trq^{(\beta)},$$

$$\frac{dq_D^{(\beta)}}{dz} = A_\alpha\varepsilon_D + C_\alpha\frac{d\varepsilon_D}{dz} + \sum_{\beta=1}^{\nu} A_{\alpha\beta}q_D^{(\beta)}, \tag{A.20}$$

where the A's, B's, C's and D's are constants.

From (A.16), proceeding as before, we again obtain expressions for σ_D and $tr\sigma$ of the forms (A.16) where $\mu(z-z')$ and $\kappa(z-z')$ still have the forms (A.17), and μ_β and $\kappa_\beta(\beta = 1, \ldots, \nu)$ are (different) constants. ρ_β and $\lambda_\beta(\beta = 1, \ldots, \nu)$ are still the solutions of eqns (A.9) and (A.12). We note, however, that in obtaining (A.17) from (A.20) we must either *assume* that $f^{(\alpha)}()$ in (A.3) depends linearly on $d\varepsilon/dz$, or that it is a sufficiently smooth function of $d\varepsilon/dz$ and that $d\varepsilon/dz$ is sufficiently small. The assumption that ε and $q^{(\beta)}$ are small does not, of course, guarantee that $d\varepsilon/dz$ is small. Indeed, $d\varepsilon/dz$ will not, in general, be small. For example, if the deformation considered is a simple shear, then the shear component of $d\varepsilon/dz$ is $(2k_2)^{-1/2}$, while the remaining components are zero.

Even when modified by the inclusion of $d\varepsilon/dz$ as an independent variable, the evolution eqn (A.19) still implies that infinitesimal changes in the internal variables resulting from a specified infinitesimal change of strain are independent of whether the strain is purely elastic or plastic. It is difficult to see what physical identification of the internal variables could lead to such a result.